has dimension $k - 1$, whereas the kernel of $d(\partial g)_s$ has dimension $k - 2$. Because this fact contradicts the conclusion we just deduced, it must be that 0 is a regular value for g. Finally, the following lemma completes the proof.

Lemma. Suppose that S is a manifold without boundary and that $\pi : S \longrightarrow \mathbf{R}$ is a smooth function with regular value 0. Then the subset $\{s \in S : \pi(s) \geq 0\}$ is a manifold with boundary, and the boundary is $\pi^{-1}(0)$.

Proof. The set where $\pi > 0$ is open in S and is therefore a submanifold of the same dimension as S. So suppose that $\pi(s) = 0$. Because π is regular at s, it is locally equivalent to the canonical submersion near s. But the lemma is obvious for the canonical submersion. Q.E.D.

The lemma is not without independent interest. For example, setting $S = \mathbf{R}^n$ and $\pi(s) = 1 - |s|^2$, it proves that the closed unit ball $\{s \in \mathbf{R}^n : |s| \leq 1\}$ is a manifold with boundary.

The generalization of Sard's theorem to manifolds with boundary is more straightforward.

Sard's theorem. For any smooth map f of a manifold X with boundary into a boundaryless manifold Y, almost every point of Y is a regular value of both $f : X \longrightarrow Y$ and $\partial f : \partial X \longrightarrow Y$.

Proof. Because the derivative of ∂f at a point $x \in \partial X$ is just the restriction of df_x to the subspace $T_x(\partial X) \subset T_x(X)$, it is obvious that if ∂f is regular at x, so is f. Thus a point $y \in Y$ fails to be a regular value of both $f : X \longrightarrow Y$ and $\partial f : \partial X \longrightarrow Y$ only when it is a critical value of $f : \text{Int}(X) \longrightarrow Y$ or $\partial f : \partial X \longrightarrow Y$. But since Int (X) and ∂X are both boundaryless manifolds, both sets of critical values have measure zero. Thus the complement of the set of common regular values for f and ∂f, being the union of two sets of measure zero, itself has measure zero. Q.E.D.

EXERCISES

1. If $U \subset \mathbf{R}^k$ and $V \subset H^k$ are neighborhoods of 0, prove that there exists no diffeomorphism of V with U.

2. Prove that if $f : X \longrightarrow Y$ is a diffeomorphism of manifolds with boundary, then ∂f maps ∂X diffeomorphically onto ∂Y.

3. Show that the square $S = [0, 1] \times [0, 1]$ is not a manifold with boundary. [HINT: If f maps a neighborhood of the corner s into H^2, and carries boundary to boundary, show that two independent vectors v_1 and v_2 in $T_s(S)$ are mapped to dependent vectors $df_s(v_1)$, $df_s(v_2)$.] See Figure 2-4.

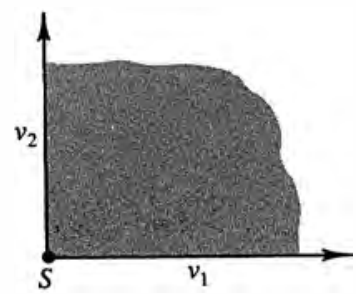

Figure 2-4

4. Show that the solid hyperboloid $x^2 + y^2 - z^2 \leq a$ is a manifold with boundary ($a > 0$).

5. Indicate for which values of a the intersection of the solid hyperboloid $x^2 + y^2 - z^2 \leq a$ and the unit sphere $x^2 + y^2 + z^2 = 1$ is a manifold with boundary? What does it look like?

6. There are two standard ways of making manifolds with boundary out of the unit square by gluing a pair of opposite edges (Figure 2-5). Simple gluing produces the cylinder, whereas gluing after one twist produces the closed Möbius band. Check that the boundary of the cylinder is two copies of S^1, while the boundary of the Möbius band is one copy of S^1; consequently, the cylinder and Möbius band are not diffeomorphic. What happens if you twist n times before gluing?

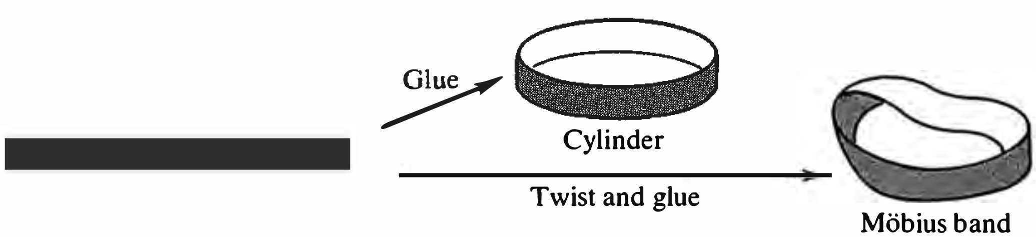

Figure 2-5

*7. Suppose that X is a manifold with boundary and $x \in \partial X$. Let $U \xrightarrow{\phi} X$ be a local parametrization with $\phi(0) = x$, where U is an open subset of H^k. Then $d\phi_0 : \mathbf{R}^k \longrightarrow T_x(X)$ is an isomorphism. Define the *upper half-space* $H_x(X)$ in $T_x(X)$ to be the image of H^k under $d\phi_0$, $H_x(X) = d\phi_0(H^k)$. Prove that $H_x(X)$ does not depend on the choice of local parametrization.

*8. Show that there are precisely two unit vectors in $T_x(X)$ that are perpendicular to $T_x(\partial X)$ and that one lies inside $H_x(X)$, the other outside. The one in $H_x(X)$ is called the *inward unit normal vector* to the boundary, and the other is the *outward unit normal vector* to the boundary. Denote the outward unit normal by $\vec{n}(x)$. Note that if X sits in $\mathbf{R}^N$, $\vec{n}$ may be considered to be a map of ∂X into $\mathbf{R}^N$. Prove that $\vec{n}$ is smooth. (Specifically, what is $\vec{n}(x)$ for $X = H^k$?) See Figure 2-6.

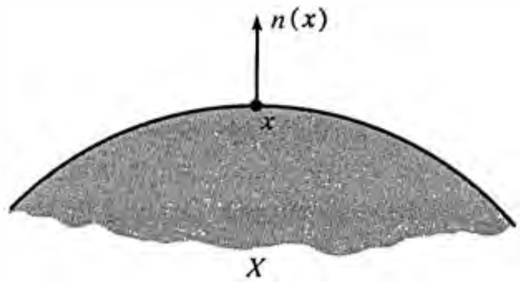

Figure 2-6

9. (a) Show that ∂X is a closed subset of X. (In particular, ∂X is compact if X is.)
   (b) Find some examples in which ∂X is compact but X is not.

10. Let $x \in \partial X$ be a boundary point. Show that there exists a smooth nonnegative function f on some open neighborhood U of x, such that $f(z) = 0$ if and only if $z \in \partial U$, and if $z \in \partial U$, then $df_z(\vec{n}(z)) > 0$.

11. (Converse to Lemma of page 62) Show that if X is any manifold with boundary, then there exists a smooth nonnegative function f on X, with a regular value at 0, such that $\partial X = f^{-1}(0)$. [HINT: Use a partition of unity to glue together the local functions of Exercise 10. What guarantees regularity?]

§2 One-Manifolds and Some Consequences

Up to diffeomorphism, the only compact, connected, one-dimensional manifolds with boundary are the closed interval and the circle. This is one of those absolutely obvious statements whose proof turns out to be technically less trivial than expected. The idea is simple enough. Beginning at some particular point, just run along the curve at constant speed. Since the manifold is compact, you cannot run forever over new territory; either you arrive again at your starting point and the curve must be a circle, or else you run into a boundary point and it is an interval. A careful proof has been provided in an appendix, so for the moment we simply assert

The Classification of One-Manifolds. Every compact, connected, one-dimensional manifold with boundary is diffeomorphic to $[0, 1]$ or S^1.